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A GLOBAL STABILITY ESTIMATE FOR THE GEL'FAND-CALDERÓN INVERSE PROBLEM IN TWO DIMENSIONS

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Abstract. We prove a global logarithmic stability estimate for the Gel'fand-Calderón inverse problem on a two-dimensional domain.

1. Introduction

Let D be an open bounded domain in $\mathbb{R}^2$ with with C^2 boundary and let $v \in C^1(\bar{D})$. The Dirichlet-to-Neumann map associated to v is the operator $\Phi : C^1(\partial D) \to L^p(\partial D),\ p < \infty$ defined by:

$$\Phi(f) = \frac{\partial u}{\partial \nu}\Big|_{\partial D} \tag{1.1}$$

where $f \in C^1(\partial D)$, ν is the outer normal of ∂D and u is the $H^1(\bar{D})$-solution of the Dirichlet problem

$$-\Delta u + v(x)u = 0 \ \text{ on } D, \ \ u|_{\partial D} = f; \tag{1.2}$$

here we assume that 0 is not a Dirichlet eigenvalue for the operator $-\Delta + v$ in D.

Equation (1.2) arises, in particular, in quantum mechanics, acoustics, electrodynamics; formally, it looks like the Schrödinger equation with potential v at zero energy.

The following inverse boundary value problem arises from this construction: given Φ on ∂D, find v on D.

This problem can be considered as the Gel'fand inverse boundary value problem for the Schrödinger equation at zero energy (see [4], [9]) and can also be seen as a generalization of the Calderón problem for the electrical impedance tomography (see [3], [9]).

The global injectivity of the map $v \to \Phi$ was firstly proved in [9] for $D \subset \mathbb{R}^d$ with $d \geq 3$ and in [2] for $d = 2$ with $v \in L^p$. A global stability estimate for the Gel'fand-Calderón problem for $d \geq 3$ was firstly proved by Alessandrini in [1]; this result was recently improved in [10].

In this paper we show that, also in the two dimensional case, an estimate of the same type as in [1] is valid. Indeed out main theorem is the following:

Theorem 1.1. *Let $D \subset \mathbb{R}^2$ be an open bounded domain with C^2 boundary, let $v_1, v_2 \in C^2(\bar{D})$ with $\|v_j\|_{C^2(\bar{D})} \le N$ for $j = 1, 2$, and Φ_1, Φ_2 the corresponding Dirichlet-to-Neumann operators. For simplicity we assume also that $v_j|_{\partial D} = 0$ and $\frac{\partial}{\partial \nu} v_j|_{\partial D} = 0$ for $j = 1, 2$. Then there exists a constant $C = C(D, N)$ such that*

(1.3)

$$\|v_2 - v_1\|_{L^\infty(D)} \le C(\log(3 + \|\Phi_2 - \Phi_1\|^{-1}))^{-\frac{1}{2}} \log(3 \log(3 + \|\Phi_2 - \Phi_1\|^{-1})),$$

where $\|A\|$ denotes the norm of an operator $A : L^\infty(\partial D) \to L^\infty(\partial D)$.

This is the first result about the global stability of the Gel'fand-Calderón inverse problem in two dimension, for general potentials. Results of such a type were only known for special kinds of potentials, e.g. potentials coming from conductivities (see [6] for example). Note also that for the Calderón problem (of the electrical impedance tomography) in its initial formulation the global injectivity was firstly proved in [11] for $d \ge 3$ and in [8] for $d = 2$.

Instability estimates complementing the stability estimates of [1], [6], [10] and of the present work are given in [7].

The proof of Theorem 1.1 takes inspiration mostly from [2] and [1]. For $z_0 \in D$ we show existence and uniqueness of a family of solution $\psi_{z_0}(z, \lambda)$ of equation (1.2) where in particular $\psi_{z_0} \to e^{\lambda(z - z_0)^2}$, for $\lambda \to \infty$. This is accomplished by introducing a special Green's function for the Laplacian which satisfies precise estimates. Then, using Alessandrini's identity along with stationary phase techniques, we obtain the result.

An extension of Theorem 1.1 for the case when we do not assume that $v_j|_{\partial D} = 0$ and $\frac{\partial}{\partial \nu} v_j|_{\partial D} = 0$ for $j = 1, 2$ is given in section 6.

2. Bukhgeim-type analogues of the Faddeev functions

In this section we introduce the above-mentioned family of solutions of equation (1.2), which will be used throughout all the paper.

We identify $\mathbb{R}^2$ with $\mathbb{C}$ and use the coordinates $z = x_1 + ix_2$, $\bar{z} = x_1 - ix_2$ where $(x_1, x_2) \in \mathbb{R}^2$. Let us define the function spaces $C^1_{\bar{z}}(\bar{D}) = \{u : u, \frac{\partial u}{\partial \bar{z}} \in C(\bar{D})\}$ with the norm $\|u\|_{C^1_{\bar{z}}(\bar{D})} = \max(\|u\|_{C(\bar{D})}, \|\frac{\partial u}{\partial \bar{z}}\|_{C(\bar{D})})$, $C^1_z(\bar{D}) = \{u :$

$u, \frac{\partial u}{\partial z} \in C(\bar{D})\}$ with an analogous norm and the following functions:

$$G_{z_0}(z,\zeta,\lambda) = e^{\lambda(z-z_0)^2} g_{z_0}(z,\zeta,\lambda) e^{-\lambda(\zeta-z_0)^2}, \tag{2.1}$$

$$g_{z_0}(z,\zeta,\lambda) = \frac{e^{\lambda(\zeta-z_0)^2-\bar{\lambda}(\bar{\zeta}-\bar{z}_0)^2}}{4\pi^2} \int_D \frac{e^{-\lambda(\eta-z_0)^2+\bar{\lambda}(\bar{\eta}-\bar{z}_0)^2}}{(z-\eta)(\bar{\eta}-\bar{\zeta})} d\mathrm{Re}\eta\, d\mathrm{Im}\eta, \tag{2.2}$$

$$\psi_{z_0}(z,\lambda) = e^{\lambda(z-z_0)^2}\mu_{z_0}(z,\lambda), \tag{2.3}$$

$$\mu_{z_0}(z,\lambda) = 1 + \int_D g_{z_0}(z,\zeta,\lambda)v(\zeta)\mu_{z_0}(\zeta,\lambda) d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta, \tag{2.4}$$

$$h_{z_0}(\lambda) = \int_D e^{\lambda(z-z_0)^2-\bar{\lambda}(\bar{z}-\bar{z}_0)^2} v(z)\mu_{z_0}(z,\lambda) d\mathrm{Re}z\, d\mathrm{Im}z, \tag{2.5}$$

where $z, z_0, \zeta \in D$ and $\lambda \in \mathbb{C}$. In addition, equation (2.4) at fixed z_0 and λ, is considered as a linear integral equation for $\mu_{z_0}(\cdot,\lambda) \in C^1_{\bar{z}}(\bar{D})$.

We have that

$$4\frac{\partial^2}{\partial z \partial \bar{z}} G_{z_0}(z,\zeta,\lambda) = \delta(z-\zeta), \tag{2.6}$$

$$4\left(\frac{\partial}{\partial z} + 2\lambda(z-z_0)\right)\frac{\partial}{\partial \bar{z}} g_{z_0}(z,\zeta,\lambda) = \delta(z-\zeta), \tag{2.7}$$

$$-4\frac{\partial^2}{\partial z \partial \bar{z}} \psi_{z_0}(z,\lambda) + v(z)\psi_{z_0}(z,\lambda) = 0, \tag{2.8}$$

$$-4\left(\frac{\partial}{\partial z} + 2\lambda(z-z_0)\right)\frac{\partial}{\partial \bar{z}} \mu_{z_0}(z,\lambda) + v(z)\mu_{z_0}(z,\lambda) = 0, \tag{2.9}$$

where $z, z_0, \zeta \in D$, $\lambda \in \mathbb{C}$, δ is the Dirac's delta. Formulas (2.6)-(2.9) follow from (2.1)-(2.4) and from

$$\frac{\partial}{\partial \bar{z}}\frac{1}{\pi z} = \delta(z), \quad \left(\frac{\partial}{\partial z} + 2\lambda(z-z_0)\right)\frac{e^{-\lambda(z-z_0)^2+\bar{\lambda}(\bar{z}-\bar{z}_0)^2}}{\pi \bar{z}} e^{\lambda z_0^2 - \bar{\lambda}\bar{z}_0^2} = \delta(z),$$

where $z, z_0, \lambda \in \mathbb{C}$.

We say that the functions G_{z_0}, g_{z_0}, ψ_{z_0}, μ_{z_0}, h_{z_0} are the Bukhgeim-type analogues of the Faddeev functions (see [9], [8], [2]).

3. Estimates for $g_{z_0}, \mu_{z_0}, h_{z_0}$

This section is devoted to crucial estimates concerning the functions defined in section 2.

Let

$$g_{z_0,\lambda}u(z) = \int_D g_{z_0}(z,\zeta,\lambda)u(\zeta) d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta, \ z \in \bar{D}, \ z_0, \lambda \in \mathbb{C}, \tag{3.1}$$

where $g_{z_0}(z,\zeta,\lambda)$ is defined by (2.2) and u is a test function.

Lemma 3.1. *Let $g_{z_0,\lambda}u$ be defined by (3.1), where $u \in C^1_{\bar z}(\bar D)$, $z_0, \lambda \in \mathbb{C}$. Then the following estimates hold:*

$$g_{z_0,\lambda}u \in C^1_{\bar z}(\bar D),$$

$$\|g_{z_0,\lambda}u\|_{C^1_{\bar z}(\bar D)} \le \frac{c_1(D)}{|\lambda|^{\frac{1}{2}}}\|u\|_{C^1_{\bar z}(\bar D)},\ |\lambda| \ge 1, \tag{3.2}$$

$$\|\frac{\partial}{\partial z} g_{z_0,\lambda}u\|_{L^p(\bar D)} \le \frac{c_2(D,p)}{|\lambda|^{\frac{1}{2}}}\|u\|_{C^1_{\bar z}(\bar D)},\ |\lambda| \ge 1,\ 1 < p < \infty. \tag{3.3}$$

Lemma 3.1 is proved in section 5.

Given a potential $v \in C^1_{\bar z}(\bar D)$ we define the operator $g_{z_0,\lambda}v$ simply as $(g_{z_0,\lambda}v)u(z) = g_{z_0,\lambda}w(z),\ w = vu$, for a test function u. If $u \in C^1_{\bar z}(\bar D)$, by Lemma 3.1 we have that $g_{z_0,\lambda}v : C^1_{\bar z}(\bar D) \to C^1_{\bar z}(\bar D)$,

$$\|g_{z_0,\lambda}v\|^{op}_{C^1_{\bar z}(\bar D)} \le 2\|g_{z_0,\lambda}\|^{op}_{C^1_{\bar z}(\bar D)}\|v\|_{C^1_{\bar z}(\bar D)}, \tag{3.4}$$

where $\|\cdot\|^{op}_{C^1_{\bar z}(\bar D)}$ denotes the operator norm in $C^1_{\bar z}(\bar D)$, $z_0, \lambda \in \mathbb{C}$. In addition, $\|g_{z_0,\lambda}\|^{op}_{C^1_{\bar z}(\bar D)}$ is estimated in Lemma 3.1. Inequality (3.4) and Lemma 3.1 implies existence and uniqueness of $\mu_{z_0}(z,\lambda)$ (and thus also $\psi_{z_0}(z,\lambda)$) for $|\lambda|$ sufficiently large.

Let

$$\mu^{(k)}_{z_0}(z,\lambda) = \sum_{j=0}^{k}(g_{z_0,\lambda}v)^j 1,$$

$$h^{(k)}_{z_0}(\lambda) = \int_D e^{\lambda(z-z_0)^2 - \bar\lambda(\bar z - \bar z_0)^2} v(z)\mu^{(k)}_{z_0}(z,\lambda) d\mathrm{Re} z\, d\mathrm{Im} z,$$

where $z, z_0 \in D$, $\lambda \in \mathbb{C}$, $k \in \mathbb{N} \cup \{0\}$.

Lemma 3.2. *For $v \in C^1_{\bar z}(\bar D)$ such that $v|_{\partial D} = 0$ the following formula holds:*

$$v(z_0) = \frac{2}{\pi}\lim_{\lambda\to\infty}|\lambda| h^{(0)}_{z_0}(\lambda),\ z_0 \in D. \tag{3.5}$$

In addition, if $v \in C^2(\bar D)$, $v|_{\partial D} = 0$ and $\frac{\partial v}{\partial \nu}|_{\partial D} = 0$ then

$$|v(z_0) - \frac{2}{\pi}|\lambda| h^{(0)}_{z_0}(\lambda)| \le c_3(D)\frac{\log(3|\lambda|)}{|\lambda|}\|v\|_{C^2(\bar D)}, \tag{3.6}$$

for $z_0 \in D$, $\lambda \in \mathbb{C}$, $|\lambda| \ge 1$.

Lemma 3.2 is proved in section 5.

Let

$$W_{z_0}(\lambda) = \int_D e^{\lambda(z-z_0)^2 - \bar\lambda(\bar z - \bar z_0)^2} w(z) d\mathrm{Re}\, z d\mathrm{Im}\, z,$$

where $z_0 \in \bar D$, $\lambda \in \mathbb{C}$ and w is some function on $\bar D$. (One can see that $W_{z_0} = h^{(0)}_{z_0}$ for $w = v$.)

Lemma 3.3. *For $w \in C^1_{\bar z}(\bar D)$ the following estimate holds:*

$$|W_{z_0}(\lambda)| \le c_4(D)\frac{\log(3|\lambda|)}{|\lambda|}\|w\|_{C^1_{\bar z}(\bar D)},\ z_0 \in \bar D,\ |\lambda| \ge 1, \tag{3.7a}$$

$$|W_{z_0}(\lambda)| \le c_{4,1}(D)\frac{\log(3|\lambda|)}{|\lambda|}\|w\|_{C(\bar D)} + \frac{c_{4,2}(D,p)}{|\lambda|}\|\frac{\partial}{\partial z}w\|_{L^p(\bar D)}, \tag{3.7b}$$

for $2 < p < \infty$.

Lemma 3.3 is proved in Section 5.

Lemma 3.4. *For $v \in C^1_{\bar z}(\bar D)$ and for $\|g_{z_0,\lambda}v\|^{op}_{C^1_{\bar z}(\bar D)} \le \delta < 1$ we have that*

$$\|\mu_{z_0}(\cdot,\lambda) - \mu^{(k)}_{z_0}(\cdot,\lambda)\|_{C^1_{\bar z}(\bar D)} \le \frac{\delta^{k+1}}{1-\delta}, \tag{3.8}$$

$$|h_{z_0}(\lambda) - h^{(k)}_{z_0}(\lambda)| \le c_4(D)\frac{\log(3|\lambda|)}{|\lambda|}\frac{\delta^{k+1}}{1-\delta}\|v\|_{C^1_{\bar z}(\bar D)}, \tag{3.9}$$

where $z_0 \in D \setminus \{0\},\ \lambda \in \mathbb{C},\ |\lambda| \ge 1,\ k \in \mathbb{N} \cup \{0\}$.

Lemma 3.4 is proved in section 5.

4. Proof of Theorem 1.1

We start from Alessandrini's identity

$$\int_D (v_2(z) - v_1(z))\psi_2(z)\psi_1(z)d\mathrm{Re}z\, d\mathrm{Im}z = \int_{\partial D}\int_{\partial D} \psi_1(z)(\Phi_2 - \Phi_1)(z,\zeta)\psi_2(\zeta)|d\zeta||dz|,$$

which holds for every ψ_j solution of $(-\Delta + v_j)\psi_j = 0$ on D, $j = 1,2$. Here $(\Phi_2 - \Phi_1)(z,\zeta)$ is the kernel of the operator $\Phi_2 - \Phi_1$.

Let $\bar\mu_{z_0}$ denote the complex conjugated of μ_{z_0} for real-valued v and, more generally, the solution of (2.4) with $g_{z_0}(z,\zeta,\lambda)$ replaced by $\overline{g_{z_0}(z,\zeta,\lambda)}$for complex-valued v. Put $\psi_1(z) = \bar\psi_{1,z_0}(z,-\lambda) = e^{-\bar\lambda(\bar z - \bar z_0)^2}\bar\mu_1(z,-\lambda)$, $\psi_2(z) = \psi_{2,z_0}(z,\lambda) = e^{\lambda(z-z_0)^2}\mu_2(z,\lambda)$, where we called for simplicity $\bar\mu_1 = \bar\mu_{1,z_0},\ \mu_2 = \mu_{2,z_0}$. This gives

$$\int_D e_{\lambda,z_0}(z)(v_2(z) - v_1(z))\mu_2(z,\lambda)\bar\mu_1(z,\lambda)d\mathrm{Re}z\, d\mathrm{Im}z = \int_{\partial D}\int_{\partial D} e^{-\bar\lambda(\bar z - \bar z_0)^2}\bar\mu_1(z,-\lambda)(\Phi_2 - \Phi_1)(z,\zeta)e^{\lambda(\zeta - z_0)^2}\mu_2(\zeta,\lambda)|d\zeta||dz|, \tag{4.1}$$

where $e_{\lambda,z_0}(z) = e^{\lambda(z-z_0)^2-\bar\lambda(\bar z-\bar z_0)^2}$. The left side $I(\lambda)$ of (4.1) can be written as the sum of four integrals, namely

$$I_1(\lambda) = \int_D e_{\lambda,z_0}(z)(v_2(z)-v_1(z))d\mathrm{Re}z\, d\mathrm{Im}z,$$
$$I_2(\lambda) = -\int_D e_{\lambda,z_0}(z)(v_2(z)-v_1(z))(\mu_2-1)(\bar\mu_1-1)d\mathrm{Re}z\, d\mathrm{Im}z,$$
$$I_3(\lambda) = -I_2(\lambda)+\int_D e_{\lambda,z_0}(z)(v_2(z)-v_1(z))(\mu_2-1)d\mathrm{Re}z\, d\mathrm{Im}z,$$
$$I_4(\lambda) = -I_2(\lambda)+\int_D e_{\lambda,z_0}(z)(v_2(z)-v_1(z))(\bar\mu_1-1)d\mathrm{Re}z\, d\mathrm{Im}z,$$

for $z_0 \in D$. By Lemma 3.1, 3.2, 3.3, 3.4 we have the following estimates:

$$\left|\frac{2}{\pi}|\lambda|I_1 - (v_2(z_0)-v_1(z_0))\right| \le c_3(D)\frac{\log(3|\lambda|)}{|\lambda|}\|v_2-v_1\|_{C^2(\bar D)}, \tag{4.2}$$

$$|I_2| \le c_5(D)\frac{\log(3|\lambda|)}{|\lambda|^{\frac{3}{2}}}\|v_2-v_1\|_{C^1(\bar D)}\|v_1\|_{C^1_z(\bar D)}\|v_2\|_{C^1_{\bar z}(\bar D)}, \tag{4.3}$$

$$|I_3| \le |I_2| + c_6(D)\frac{\log(3|\lambda|)}{|\lambda|^{\frac{3}{2}}}\|v_2-v_1\|_{C^1_{\bar z}(\bar D)}\|v_2\|_{C^1_{\bar z}(\bar D)}, \tag{4.4}$$

$$|I_4| \le |I_2| + c_6(D)\frac{\log(3|\lambda|)}{|\lambda|^{\frac{3}{2}}}\|v_2-v_1\|^2_{C^1_z(\bar D)}\|v_1\|_{C^1_z(\bar D)}, \tag{4.5}$$

for $|\lambda|$ sufficiently large for example, for λ such that

$$\frac{2\,c_1(D)}{|\lambda|^{\frac{1}{2}}}\max\left(\|v_1\|_{C^1_{\bar z}(\bar D)}, \|v_1\|_{C^1_z(\bar D)}, \|v_2\|_{C^1_{\bar z}(\bar D)}, \|v_2\|_{C^1_z(\bar D)}\right) \le \frac{1}{2},\ |\lambda| \ge 1. \tag{4.6}$$

The right side $J(\lambda)$ of (4.1) can be estimated as follows:

$$|\lambda||J(\lambda)| \le c_7(D)e^{(2L^2+1)|\lambda|}\|\Phi_2-\Phi_1\|, \tag{4.7}$$

where we called $L = \max_{z\in\partial D,\, z_0\in D}|z-z_0|$.

Putting together estimates (4.2)-(4.7) we obtain

$$|v_2(z_0)-v_1(z_0)| \le c_8(D)\frac{\log(3|\lambda|)}{|\lambda|^{\frac{1}{2}}}N^3 + \frac{2}{\pi}c_7(D)e^{(2L^2+1)|\lambda|}\|\Phi_2-\Phi_1\| \tag{4.8}$$

for $z_0 \in D$ and N is the costant in the statement of Theorem 1.1. We call $\varepsilon = \|\Phi_2-\Phi_1\|$ and impose $|\lambda| = \gamma\log(3+\varepsilon^{-1})$, where $0<\gamma<(2L^2+1)^{-1}$ so that (4.8) reads

$$\begin{aligned}|v_2(z_0)-v_1(z_0)| \le{}& c_8(D)N^3(\gamma\log(3+\varepsilon^{-1}))^{-\frac{1}{2}}\log(3\gamma\log(3+\varepsilon^{-1}))\\ &+\frac{2}{\pi}c_7(D)(3+\varepsilon^{-1})^{(2L^2+1)\gamma}\varepsilon,\end{aligned} \tag{4.9}$$

for every $z_0 \in D$, with

$$0 < \varepsilon \leq \varepsilon_1(D, N, \gamma), \tag{4.10}$$

where ε_1 is sufficiently small or, more precisely, where (4.10) implies that $|\lambda| = \gamma \log(3 + \varepsilon^{-1})$ satisfies (4.6).

As $(3 + \varepsilon^{-1})^{(2L^2+1)\gamma}\varepsilon \to 0$ for $\varepsilon \to 0$ more rapidly then the other term, we obtain that

$$\|v_2 - v_1\|_{L^\infty(D)} \leq c_9(D, N, \gamma) \frac{\log(3 \log(3 + \|\Phi_2 - \Phi_1\|^{-1}))}{(\log(3 + \|\Phi_2 - \Phi_1\|^{-1}))^{\frac{1}{2}}} \tag{4.11}$$

for $\varepsilon = \|\Phi_2 - \Phi_1\| \leq \varepsilon_1(D, N, \gamma)$.

Estimate (4.11) for general ε (with modified c_{10}) follows from (4.11) for $\varepsilon \leq \varepsilon_1(D, N, \gamma)$ and the assumption that $\|v_j\|_{L^\infty(D)} \leq N$, $j = 1, 2$. This completes the proof of Theorem 1.1.

5. Proofs of the Lemmata

Proof of Lemma 3.1. One can see that $g_{z_0,\lambda} = \frac{1}{4} T \bar{T}_{z_0,\lambda}$, for $z_0, \lambda \in \mathbb{C}$, where

(5.1)
$$Tu(z) = -\frac{1}{\pi} \int_D \frac{u(\zeta)}{\zeta - z} d\mathrm{Re}\zeta \, d\mathrm{Im}\zeta,$$

(5.2)
$$\bar{T}_{z_0,\lambda} u(z) = -\frac{e^{-\lambda(z-z_0)^2 + \bar{\lambda}(\bar{z}-\bar{z}_0)^2}}{\pi} \int_D \frac{e^{\lambda(\zeta - z_0)^2 - \bar{\lambda}(\bar{\zeta} - \bar{z}_0)^2}}{\bar{\zeta} - \bar{z}} u(\zeta) d\mathrm{Re}\zeta \, d\mathrm{Im}\zeta,$$

for $z \in \bar{D}$ and u a test function. Estimates (3.2), (3.3) now follow from

$$Tw \in C^1_{\bar{z}}(\bar{D}), \tag{5.3}$$

$$\|Tw\|_{C^1_{\bar{z}}(\bar{D})} \leq n_1(D) \|w\|_{C(\bar{D})}, \text{ where } w \in C(D), \tag{5.4a}$$

$$\|\frac{\partial T}{\partial z} w\|_{L^p(\bar{D})} \leq n(D, p) \|w\|_{L^p(\bar{D})}, \ 1 < p < \infty, \tag{5.4b}$$

$$\bar{T}_{z_0,\lambda} u \in C(\bar{D}), \tag{5.5}$$

$$\|\bar{T}_{z_0,\lambda} u\|_{C(\bar{D})} \leq \frac{n_2(D)}{|\lambda|^{\frac{1}{2}}} \|u\|_{C^1_{\bar{z}}(\bar{D})}, \ |\lambda| \geq 1, \tag{5.6}$$

$$\|\bar{T}_{z_0,\lambda} u\|_{C(\bar{D})} \leq \frac{\log(3|\lambda|)(1 + |z - z_0|) n_3(D)}{|\lambda||z - z_0|^2} \|u\|_{C^1_{\bar{z}}(\bar{D})}, \ |\lambda| \geq 1, \tag{5.7}$$

where $u \in C^1_{\bar{z}}(\bar{D})$, $z_0, \lambda \in \mathbb{C}$. Estimates (5.3), (5.4) are well-known (see [12]).

The assumption $u \in C^1_{\bar z}(\bar D)$ is not necessary at all for (5.5): indeed, using well-known arguments it is sufficient to take $u \in C(\bar D)$.

Let us prove (5.6) and (5.7). We have that

$$-\pi e^{\lambda(z-z_0)^2-\bar\lambda(\bar z-\bar z_0)^2}\bar T_{z_0,\lambda}u(z) = I_{z_0,\lambda,\varepsilon}(z) + J_{z_0,\lambda,\varepsilon}(z),$$

where

$$I_{z_0,\lambda,\varepsilon}(z) = \int_{D\cap(B_{z,\varepsilon}\cup B_{z_0,\varepsilon})} \frac{e^{\lambda(\zeta-z_0)^2-\bar\lambda(\bar\zeta-\bar z_0)^2}}{\bar\zeta-\bar z}u(\zeta)d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta, \tag{5.8}$$

$$J_{z_0,\lambda,\varepsilon}(z) = \int_{D_{z,z_0,\varepsilon}} \frac{e^{\lambda(\zeta-z_0)^2-\bar\lambda(\bar\zeta-\bar z_0)^2}}{\bar\zeta-\bar z}u(\zeta)d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta, \tag{5.9}$$

and $B_{z,\varepsilon} = \{\zeta \in \mathbb{C} : |\zeta - z| < \varepsilon\}$, $D_{z,z_0,\varepsilon} = D \setminus (B_{z,\varepsilon} \cup B_{z_0,\varepsilon})$. One sees that

$$|I_{z_0,\lambda,\varepsilon}(z)| \le 2\int_{B_{z,\varepsilon}} \frac{\|u\|_{C(\bar D)}}{|\zeta-z|}d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta = 4\pi\varepsilon\|u\|_{C(\bar D)}, \tag{5.10}$$

with $z, z_0, \lambda \in \mathbb{C}$, $\varepsilon > 0$. Further, we have that

$$\begin{aligned} J_{z_0,\lambda,\varepsilon}(z) &= -\frac{1}{2\bar\lambda}\int_{D_{z,z_0,\varepsilon}} \frac{\partial}{\partial\bar\zeta}\left(e^{\lambda(\zeta-z_0)^2-\bar\lambda(\bar\zeta-\bar z_0)^2}\right)\frac{u(\zeta)}{(\bar\zeta-\bar z)(\bar\zeta-\bar z_0)}d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta \\ &= J^1_{z_0,\lambda,\varepsilon}(z) + J^2_{z_0,\lambda,\varepsilon}(z), \end{aligned}$$

where

$$\begin{aligned} J^1_{z_0,\lambda,\varepsilon}(z) &= -\frac{1}{4i\bar\lambda}\int_{\partial D_{z,z_0,\varepsilon}} \frac{e^{\lambda(\zeta-z_0)^2-\bar\lambda(\bar\zeta-\bar z_0)^2}}{(\bar\zeta-\bar z)(\bar\zeta-\bar z_0)}u(\zeta)d\zeta, \\ J^2_{z_0,\lambda,\varepsilon}(z) &= \frac{1}{2\bar\lambda}\int_{D_{z,z_0,\varepsilon}} e^{\lambda(\zeta-z_0)^2-\bar\lambda(\bar\zeta-\bar z_0)^2}\frac{\partial}{\partial\bar\zeta}\left(\frac{u(\zeta)}{(\bar\zeta-\bar z)(\bar\zeta-\bar z_0)}\right)d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta \end{aligned}$$

Now we get

$$|J^1_{z_0,\lambda,\varepsilon}(z)| \le M^1_{z,z_0,\lambda,\varepsilon} := \frac{1}{4|\lambda|}\int_{\partial D_{z,z_0,\varepsilon}} \frac{|u(\zeta)||d\zeta|}{|\bar\zeta-\bar z||\bar\zeta-\bar z_0|},$$

$$M^1_{z,z_0,\lambda,\varepsilon} \le \frac{1}{8|\lambda|}\int_{\partial D_{z,z_0,\varepsilon}}\left(\frac{1}{|\bar\zeta-\bar z|^2}+\frac{1}{|\bar\zeta-\bar z_0|^2}\right)|d\zeta|\|u\|_{C(D)}, \tag{5.11}$$

$$M^1_{z,z_0,\lambda,\varepsilon} \le \frac{1}{2|z-z_0||\lambda|}\int_{\partial D_{z,z_0,\varepsilon}}\left(\frac{1}{|\bar\zeta-\bar z|}+\frac{1}{|\bar\zeta-\bar z_0|}\right)|d\zeta|\|u\|_{C(D)}. \tag{5.12}$$

We also have

$$\begin{aligned} |J^2_{z_0,\lambda,\varepsilon}(z)| \le M^2_{z,z_0,\lambda,\varepsilon} :&= \frac{1}{2|\lambda|}\int_{D_{z,z_0,\varepsilon}} \frac{|\frac{\partial u}{\partial\bar\zeta}(\zeta)|}{|\bar\zeta-\bar z||\bar\zeta-\bar z_0|} + \frac{|u(\zeta)|}{|\bar\zeta-\bar z|^2|\bar\zeta-\bar z_0|} \\ &\quad + \frac{|u(\zeta)|}{|\bar\zeta-\bar z||\bar\zeta-\bar z_0|^2}d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta, \end{aligned}$$

(5.13)
$$M^2_{z,z_0,\lambda,\varepsilon} \le \frac{1}{2|\lambda|}\int_{D_{z,z_0,\varepsilon}} \frac{|\frac{\partial u}{\partial \zeta}(\zeta)|}{|\bar\zeta-\bar z|^2} + \frac{|\frac{\partial u}{\partial \zeta}(\zeta)|}{|\bar\zeta-\bar z_0|^2} + 2\frac{|u(\zeta)|}{|\bar\zeta-\bar z|^3} + 2\frac{|u(\zeta)|}{|\bar\zeta-\bar z_0|^3} d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta$$

(5.14)
$$M^2_{z,z_0,\lambda,\varepsilon} \le \frac{1}{2|\lambda|}\int_{D_{z,z_0,\varepsilon}} \frac{2|\frac{\partial u}{\partial \zeta}(\zeta)|}{|\bar\zeta-\bar z||z-z_0|} + \frac{2|\frac{\partial u}{\partial \zeta}(\zeta)|}{|\bar\zeta-\bar z_0||z-z_0|} + \frac{2|u(\zeta)|}{|\bar\zeta-\bar z|^2|z-z_0|} + \frac{4|u(\zeta)|}{|\bar\zeta-\bar z||z-z_0|^2} + \frac{2|u(\zeta)|}{|\bar\zeta-\bar z_0|^2|z-z_0|} + \frac{4|u(\zeta)|}{|\bar\zeta-\bar z_0||z-z_0|^2} d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta.$$

Using (5.11) and (5.13) we obtain that

(5.15)
$$|J^1_{z_0,\lambda,\varepsilon}(z)| \le |\lambda|^{-1} n_4(D)\varepsilon^{-1}\|u\|_{C(D)},$$

(5.16)
$$|J^2_{z_0,\lambda,\varepsilon}(z)| \le |\lambda|^{-1} n_5(D)\varepsilon^{-1}\|u\|_{C(D)} + |\lambda|^{-1} n_6(D)\log(3\varepsilon^{-1})\|\frac{\partial u}{\partial \bar z}\|_{C(D)},$$

where $z, z_0, \lambda \in \mathbb{C}$, $|\lambda| \ge 1$, $0 < \varepsilon < 1$.

If $z_0 \ne z$ we can use (5.12) and (5.14) in order to obtain

$$|J^1_{z_0,\lambda,\varepsilon}(z)| \le |\lambda|^{-1}|z-z_0|^{-1} n_7(D)\log(3\varepsilon^{-1})\|u\|_{C(D)}, \tag{5.17}$$

$$|J^2_{z_0,\lambda,\varepsilon}(z)| \le |\lambda|^{-1}|z-z_0|^{-2} n_8(D)\log(3\varepsilon^{-1})\|u\|_{C(D)} + |\lambda|^{-1}|z-z_0|^{-1} n_9(D)\|\frac{\partial u}{\partial \bar z}\|_{C(D)}, \tag{5.18}$$

Finally, putting $\varepsilon = |\lambda|^{-\frac{1}{2}}$ into (5.10), (5.15), (5.16) we obtain (5.6), while putting $\varepsilon = |\lambda|^{-1}$ into (5.10), (5.17), (5.18) we obtain (5.7). The proof follows. □

Proof of Lemma 3.2. First we extend our potential v to a larger domain $D_1 \supset D$ (always with C^2 boundary) such that $dist(\partial D_1, \partial D) \ge \delta > 0$ (for some δ) by putting $v|_{D_1\setminus D} \equiv 0$. In such a way $v \in C^1(D_1) \cap C^2(D_1 \setminus \partial D)$ with $\|v\|_{C^k(D_1)} = \|v\|_{C^k(D)}$ for $k = 1, 2$.

Now let χ_δ be a real-valued function on $\mathbb{C}$, with $\delta > 0$, constructed as follows:

$$\begin{aligned}
&\chi_\delta(z) = \chi(z/\delta), \text{ where}\\
&\chi \in C^\infty(\mathbb{C}),\ \chi \text{ is real valued},\\
&\chi(z) = \chi(|z|),\\
&\chi(z) \equiv 1 \text{ for } |z| \le 1/2,\\
&\chi(z) \equiv 0 \text{ for } |z| \ge 1.
\end{aligned}$$

Let

$$v_{lin}(z,z_0) = v(z_0) + v_z(z_0)(z-z_0) + v_{\bar z}(z_0)(\bar z - \bar z_0),$$

for $z, z_0 \in D_1$, $v_z = \frac{\partial v}{\partial z}$ and $v_{\bar z} = \frac{\partial v}{\partial \bar z}$.

We can write $h^{(0)}_{z_0}(\lambda) = S_{z_0,\delta}(\lambda) + R_{z_0,\delta}(\lambda)$, where

$$\begin{aligned} S_{z_0,\delta}(\lambda) &= \int_{\mathbb{C}} e^{\lambda(z-z_0)^2 - \bar\lambda(\bar z - \bar z_0)^2} v_{lin}(z,z_0)\chi_\delta(z-z_0) d\mathrm{Re} z\, d\mathrm{Im} z \\ &= \int_{\mathbb{C}} e^{i|\lambda|(z^2+\bar z^2)} v_{lin}(e^{-i\varphi(\lambda)}z + z_0, z_0)\chi_\delta(z) d\mathrm{Re} z\, d\mathrm{Im} z, \\ R_{z_0,\delta}(\lambda) &= \int_{D_1} e^{\lambda(z-z_0)^2 - \bar\lambda(\bar z - \bar z_0)^2} \left(v(z) - v_{lin}(z,z_0)\chi_\delta(z-z_0)\right) d\mathrm{Re} z\, d\mathrm{Im} z \end{aligned}$$

where $\varphi(\lambda) = \frac{1}{2}(\arg(\lambda) - \frac{\pi}{2})$, $z_0 \in D$, $\lambda \in \mathbb{C}$.

Using the stationary phase method we obtain that

$$v(z_0) = \frac{2}{\pi} \lim_{\lambda\to\infty} |\lambda| S_{z_0,\delta}(\lambda), \tag{5.19}$$

$$|v(z_0) - \frac{2}{\pi}|\lambda| S_{z_0,\delta}(\lambda)| \le q_1(D,\delta)\|v\|_{C^1(\bar D)} |\lambda|^{-1}, \tag{5.20}$$

$z_0 \in D$, $\delta > 0$, $\lambda \in \mathbb{C}$, $|\lambda| \ge 1$. Integrating by parts we can write

$$\begin{aligned} R_{z_0,\delta}(\lambda) &= -\frac{1}{2\bar\lambda}\int_{D_1} \frac{\partial}{\partial \bar z}\left(e^{\lambda(z-z_0)^2 - \bar\lambda(\bar z - \bar z_0)^2}\right) \\ &\quad\times \frac{(v(z) - v_{lin}(z,z_0)\chi_\delta(z-z_0))}{\bar z - \bar z_0} d\mathrm{Re} z\, d\mathrm{Im} z = R^1_{z_0,\delta}(\lambda) + R^2_{z_0,\delta}(\lambda), \\ R^1_{z_0,\delta}(\lambda) &= \frac{-1}{4i\bar\lambda}\int_{\partial D_1} e^{\lambda(z-z_0)^2 - \bar\lambda(\bar z - \bar z_0)^2} \frac{(v(z) - v_{lin}(z,z_0)\chi_\delta(z-z_0))}{\bar z - \bar z_0} dz, \\ R^2_{z_0,\delta}(\lambda) &= \frac{1}{2\bar\lambda}\int_{D_1} e^{\lambda(z-z_0)^2 - \bar\lambda(\bar z - \bar z_0)^2} \\ &\quad\times \frac{\partial}{\partial \bar z}\left(\frac{(v(z) - v_{lin}(z,z_0)\chi_\delta(z-z_0))}{\bar z - \bar z_0}\right) d\mathrm{Re} z\, d\mathrm{Im} z, \end{aligned}$$

for $z_0 \in D$, $\lambda \in \mathbb{C}\setminus\{0\}$. In addition, we have that

$$\lim_{\lambda\to\infty} |\lambda| R^1_{z_0,\delta}(\lambda) = 0, \tag{5.21}$$

$$\lim_{\lambda\to\infty} |\lambda| R^2_{z_0,\delta}(\lambda) = 0. \tag{5.22}$$

Formula (5.21) follows from properties of χ_δ, the assumption that $z_0 \in D$ and that $v|_{\partial D_1} \equiv 0$. Actually, as a corollary of this properties we have that $v(z) - v_{lin}(z,z_0)\chi_\delta(z-z_0) \equiv 0$ for $z \in \partial D_1$ and, therefore, $R^1_{z_0,\delta}(\lambda) \equiv 0$ for $\lambda \in \mathbb{C}\setminus\{0\}$.

Formula (5.22) for $v \in C^1(\bar D_1)$ is a consequence of the estimates

$$R^{2,1}_{z_0,\delta,\varepsilon}(\lambda) := \int_{B_{z_0,\varepsilon}} e^{\lambda(z-z_0)^2-\bar\lambda(\bar z-\bar z_0)^2} \times \frac{\partial}{\partial \bar z}\left(\frac{(v(z)-v_{lin}(z,z_0)\chi_\delta(z-z_0))}{\bar z-\bar z_0}\right) d\mathrm{Re}z\, d\mathrm{Im}z = O(\varepsilon) \text{ as } \varepsilon\to 0 \tag{5.23}$$

$$R^{2,2}_{z_0,\delta,\varepsilon}(\lambda) := \int_{D_{z_0,\varepsilon}} e^{\lambda(z-z_0)^2-\bar\lambda(\bar z-\bar z_0)^2} \times \frac{\partial}{\partial \bar z}\left(\frac{(v(z)-v_{lin}(z,z_0)\chi_\delta(z-z_0))}{\bar z-\bar z_0}\right) d\mathrm{Re}z\, d\mathrm{Im}z \to 0 \text{ as } \lambda\to \infty \tag{5.24}$$

where $B_{z_0,\varepsilon}=\{z\in\mathbb{C} : |z-z_0|<\varepsilon\}$, $D_{z_0,\varepsilon}=D_1\setminus B_{z_0,\varepsilon}$. In (5.23)-(5.24) we assume that $z_0\in D,\ 0<\varepsilon<\delta,\ \lambda\in\mathbb{C}$.

Estimate (5.23) is obtained by standard arguments using that

$$|v(z)-v(z_0)| \le \|v\|_{C^1(\bar D)}|z-z_0|,\ z_0\in D,\ z\in B_{z_0,\delta},$$

while (5.24) is a variation of the Riemann-Lebesgue Lemma.

Formula (3.5) now follows from (5.19), (5.21), (5.22).

Under the assumptions mentioned in Lemma 3.2, the final part of the proof of estimate (3.6) consists in the following. We have, for $\varepsilon<\delta/2$,

$$\begin{aligned}
|R^{2,1}_{z_0,\delta,\varepsilon}(\lambda)| &\le \int_{B_{z_0,\varepsilon}} \frac{|v(z)-v_{lin}(z,z_0)|}{|z-z_0|^2} d\mathrm{Re}z\, d\mathrm{Im}z \\
&\quad + \int_{B_{z_0,\varepsilon}} \frac{|v_{\bar z}(z)-v_{\bar z}(z_0)|}{|z-z_0|} d\mathrm{Re}z\, d\mathrm{Im}z \le \frac{7}{2}\pi\|v\|_{C^2(\bar D)}\varepsilon^2, \\
R^{2,2}_{z_0,\delta,\varepsilon}(\lambda) &= \frac{-1}{2\bar\lambda}\int_{D_{z_0,\varepsilon}} \frac{\partial}{\partial\bar z}\left(e^{\lambda(z-z_0)^2-\bar\lambda(\bar z-\bar z_0)^2}\right)\frac{1}{\bar z-\bar z_0} \\
&\quad \times \frac{\partial}{\partial \bar z}\left(\frac{(v(z)-v_{lin}(z,z_0)\chi_\delta(z-z_0))}{\bar z-\bar z_0}\right) d\mathrm{Re}z\, d\mathrm{Im}z \\
&= \frac{-1}{2\bar\lambda}(R^{2,2,1}_{z_0,\delta,\varepsilon}(\lambda)+R^{2,2,2}_{z_0,\delta,\varepsilon}(\lambda)), \\
R^{2,2,1}_{z_0,\delta,\varepsilon}(\lambda) &= \frac{1}{2i}\int_{\partial D_{z_0,\varepsilon}} e^{\lambda(z-z_0)^2-\bar\lambda(\bar z-\bar z_0)^2}\frac{1}{\bar z-\bar z_0} \\
&\quad \times \frac{\partial}{\partial \bar z}\left(\frac{(v(z)-v_{lin}(z,z_0)\chi_\delta(z-z_0))}{\bar z-\bar z_0}\right) dz
\end{aligned} \tag{5.25}$$

$$= \frac{-1}{2i}\int_{\partial B_{z_0,\varepsilon}} e^{\lambda(z-z_0)^2-\bar\lambda(\bar z-\bar z_0)^2}\frac{1}{\bar z-\bar z_0}\frac{\partial}{\partial\bar z}\left(\frac{v(z)-v_{lin}(z,z_0)}{\bar z-\bar z_0}\right)dz,$$

where we used in particular that $v|_{\partial D_1} \equiv 0$, $\frac{\partial}{\partial\nu} v|_{\partial D_1} \equiv 0$,

$$R^{2,2,2}_{z_0,\delta,\varepsilon}(\lambda) = -\int_{D_{z_0,\varepsilon}} e^{\lambda(z-z_0)^2-\bar\lambda(\bar z-\bar z_0)^2} \times \frac{\partial}{\partial \bar z}\left(\frac{1}{\bar z-\bar z_0}\frac{\partial}{\partial \bar z}\left(\frac{(v(z)-v_{lin}(z,z_0)\chi_\delta(z-z_0))}{\bar z-\bar z_0}\right)\right) d\mathrm{Re}z\, d\mathrm{Im}z.$$

We have, for $\varepsilon < \delta/2$

$$|R^{2,2,1}_{z_0,\delta,\varepsilon}(\lambda)| \le \frac{1}{2}\int_{\partial B_{z_0,\varepsilon}} \frac{|v(z)-v_{lin}(z,z_0)|}{|z-z_0|^3}|dz| + \frac{1}{2}\int_{\partial B_{z_0,\varepsilon}} \frac{|v_{\bar z}(z)-v_{\bar z}(z_0)|}{|z-z_0|^2}|dz| \le \frac{7}{2}\pi\|v\|_{C^2(\bar D)}, \tag{5.26}$$

$$|R^{2,2,2}_{z_0,\delta,\varepsilon}(\lambda)| \le |R^{2,2,2}_{z_0,\delta,\delta/2}(\lambda)| + |R^{2,2,2}_{z_0,\delta,\varepsilon}(\lambda) - R^{2,2,2}_{z_0,\delta,\delta/2}(\lambda)|, \tag{5.27}$$

$$|R^{2,2,2}_{z_0,\delta,\delta/2}(\lambda)| \le q_2(D,\delta)\|v\|_{C^2(\bar D)}, \tag{5.28}$$

$$|R^{2,2,2}_{z_0,\delta,\varepsilon}(\lambda) - R^{2,2,2}_{z_0,\delta,\delta/2}(\lambda)| \le \sum_{j=1}^{5}\int_{B_{z_0,\delta/2}\setminus B_{z_0,\varepsilon}} u_j(z,z_0) d\mathrm{Re}z\, d\mathrm{Im}z,$$

with

$$u_1(z,z_0) = \frac{1}{|z-z_0|^2}\left|\frac{v_{\bar z}(z)-v_{\bar z}(z_0)}{\bar z-\bar z_0}\right|, \tag{5.29}$$

$$u_2(z,z_0) = \frac{1}{|z-z_0|^2}\left|\frac{v(z)-v_{lin}(z,z_0)}{(\bar z-\bar z_0)^2}\right|, \tag{5.30}$$

$$u_3(z,z_0) = \frac{1}{|z-z_0|}\left|\frac{v_{\bar z\bar z}(z)}{\bar z-\bar z_0}\right|, \tag{5.31}$$

$$u_4(z,z_0) = \frac{2}{|z-z_0|}\left|\frac{v_{\bar z}(z)-v_{\bar z}(z_0)}{(\bar z-\bar z_0)^2}\right|, \tag{5.32}$$

$$u_5(z,z_0) = \frac{2}{|z-z_0|}\left|\frac{v(z)-v_{lin}(z,z_0)}{(\bar z-\bar z_0)^3}\right|. \tag{5.33}$$

This yields

$$|R^{2,2,2}_{z_0,\delta,\varepsilon}(\lambda) - R^{2,2,2}_{z_0,\delta,\delta/2}(\lambda)| \le q_3 \log(\frac{\delta}{2\varepsilon})\|v\|_{C^2(\bar D)}, \tag{5.34}$$

where $z_0 \in D$, $0 < \varepsilon < \delta/2$. $\lambda \in \mathbb{C}\setminus\{0\}$. Using (5.20), (5.25)-(5.34) with $\varepsilon = |\lambda|^{-1}$ we obtain (3.6). Lemma 3.2 is proved. □

Proof of Lemma 3.3. We write

$$W_{z_0}(\lambda) = W^1_{z_0,\varepsilon}(\lambda) + W^2_{z_0,\varepsilon}(\lambda),$$
$$W^1_{z_0,\varepsilon}(\lambda) = \int_{D\cap B_{z_0,\varepsilon}} e^{\lambda(z-z_0)^2-\bar\lambda(\bar z-\bar z_0)^2} w(z) d\mathrm{Re}z\, d\mathrm{Im}z,$$
$$W^2_{z_0,\varepsilon}(\lambda) = \int_{D\setminus B_{z_0,\varepsilon}} e^{\lambda(z-z_0)^2-\bar\lambda(\bar z-\bar z_0)^2} w(z) d\mathrm{Re}z\, d\mathrm{Im}z,$$

where $B_{z_0,\varepsilon} = \{z \in \mathbb{C} : |z - z_0| < \varepsilon\}$. One sees that

$$|W^1_{z_0,\varepsilon}(\lambda)| \le \int_{D\cap B_{z_0,\varepsilon}} \|w\|_{C(D)} d\mathrm{Re}z\, d\mathrm{Im}z = \pi\|w\|_{C(D)}\varepsilon^2, \tag{5.35}$$

$$W^2_{z_0,\varepsilon}(\lambda) = \frac{-1}{2\bar\lambda}\int_{D\setminus B_{z_0,\varepsilon}} \frac{\partial}{\partial\bar z}\left(e^{\lambda(z-z_0)^2-\bar\lambda(\bar z-\bar z_0)^2}\right)\frac{w(z)}{\bar z - \bar z_0} d\mathrm{Re}z\, d\mathrm{Im}z$$
$$= W^{2,1}_{z_0,\varepsilon}(\lambda) + W^{2,2}_{z_0,\varepsilon}(\lambda),$$
$$W^{2,1}_{z_0,\varepsilon}(\lambda) = \frac{-1}{4i\bar\lambda}\int_{\partial(D\setminus B_{z_0,\varepsilon})} e^{\lambda(z-z_0)^2-\bar\lambda(\bar z-\bar z_0)^2}\frac{w(z)}{\bar z-\bar z_0}dz,$$
$$W^{2,2}_{z_0,\varepsilon}(\lambda) = \frac{1}{2\bar\lambda}\int_{D\setminus B_{z_0,\varepsilon}} e^{\lambda(z-z_0)^2-\bar\lambda(\bar z-\bar z_0)^2}\frac{\partial}{\partial\bar z}\left(\frac{w(z)}{\bar z-\bar z_0}\right) d\mathrm{Re}z\, d\mathrm{Im}z.$$

We have

$$|W^{2,1}_{z_0,\varepsilon}(\lambda)| \le |\lambda|^{-1}a_1(D)\|w\|_{C(\bar D)}\log(3\varepsilon^{-1}), \tag{5.36}$$

$$|W^{2,2}_{z_0,\varepsilon}(\lambda)| \le |\lambda|^{-1}a_2(D)\|w\|_{C^1_{\bar z}(\bar D)}\log(3\varepsilon^{-1}) \tag{5.37a}$$

$$|W^{2,2}_{z_0,\varepsilon}(\lambda)| \le |\lambda|^{-1}a_2(D)\|w\|_{C(\bar D)}\log(3\varepsilon^{-1}) + |\lambda|^{-1}a_3(D,p)\|\frac{\partial w}{\partial\bar z}\|_{L^p(\bar D)}, \tag{5.37b}$$

for $z_0 \in D,\ \lambda \in \mathbb{C}\setminus\{0\},\ 0 < \varepsilon \le 1,\ 2 < p < \infty$.

Using (5.35), (5.36), (5.37) with $\varepsilon = |\lambda|^{-1}$ we obtain (3.7). This finishes the proof. □

Proof of Lemma 3.4. Formula (3.8) follows from the assumption on $\|g_{z_0,\lambda}v\|$ and from solving (2.4) by the method of successive approximations. The proof of estimate (3.9) follows from (3.8) and Lemma 3.3. The proof follows. □

6. An extension of Theorem 1.1

As an extension of Theorem 1.1 for the case when we do not assume that $v_j|_{\partial D} \equiv 0,\ \frac{\partial}{\partial\nu}v_j|_{\partial D} \equiv 0,\ j = 1, 2$, we give the following result.

Proposition 6.1. *Let $D \subset \mathbb{R}^2$ be an open bounded domain with C^2 boundary, let $v_1, v_2 \in C^2(\bar D)$ with $\|v_j\|_{C^2(\bar D)} \le N$ for $j = 1, 2$, and Φ_1, Φ_2 the corresponding Dirichlet-to-Neumann operators. Then, for any $0 < \alpha < \frac{1}{5}$, there exists a constant $C = C(D, N, \alpha)$ such that the following inequality holds*

$$\|v_2 - v_1\|_{L^\infty(D)} \le C\log(3 + \|\Phi_2 - \Phi_1\|_1^{-1})^{-\alpha}, \tag{6.1}$$

where $\|A\|_1$ *is the norm for an operator* $A : L^\infty(\partial D) \to L^\infty(\partial D)$, *with kernel* $A(x,y)$, *defined as* $\|A\|_1 = \sup_{x,y\in\partial D} |A(x,y)|(\log(3+|x-y|^{-1}))^{-1}$.

All we need to know about $\|\cdot\|_1$ consists of the following:

i) $\|A\|_{L^\infty(\partial D)\to L^\infty(\partial D)} \le const(D)\|A\|_1$;

ii) by formula (4.9) of [9] one has

$$\|v\|_{L^\infty(\partial D)} \le const\|\Phi_v - \Phi_0\|_1.$$

In order to prove Proposition 6.1 we need the following modified version of Lemma 3.2. We will call $(\partial D)_\delta = \{z \in \mathbb{C} : dist(z, \partial D) < \delta\}$.

Lemma 6.2. *For* $v \in C^2(\bar D)$ *we have that*

(6.2)

$$|v(z_0) - \frac{2}{\pi}|\lambda| h^{(0)}_{z_0}(\lambda)| \le \kappa_1(D)\delta^{-4}\frac{\log(3|\lambda|)}{|\lambda|}\|v\|_{C^2(\bar D)} + \kappa_2(D)\log(3+\delta^{-1})\|v\|_{C(\partial D)},$$

for $z_0 \in D\setminus(\partial D)_\delta$, $0<\delta<1$, $\lambda\in\mathbb{C}$, $|\lambda|\ge 1$.

Proof of Lemma 6.2. Let χ_δ be as in the proof of Lemma 3.2. We have in particular that

$$\|\chi_\delta\|_{C^k(\mathbb{C})} \le \delta^{-k}\|\chi\|_{C^k(\mathbb{C})},\ k\in\mathbb{N}. \tag{6.3}$$

Let

$$v_{lin}(z,z_0) = v(z_0) + v_z(z_0)(z-z_0) + v_{\bar z}(z_0)(\bar z - \bar z_0),$$

for $z, z_0 \in D$, $v_z = \frac{\partial v}{\partial z}$ and $v_{\bar z} = \frac{\partial v}{\partial \bar z}$.

We can write $h^{(0)}_{z_0}(\lambda) = S_{z_0,\delta}(\lambda) + R_{z_0,\delta}(\lambda)$, where

$$\begin{aligned} S_{z_0,\delta}(\lambda) &= \int_{\mathbb{C}} e_{\lambda,z_0}(z) v_{lin}(z,z_0)\chi_\delta(z-z_0) d\mathrm{Re}z\, d\mathrm{Im}z \\ &= \int_{\mathbb{C}} e^{i|\lambda|(z^2+\bar z^2)} v_{lin}(e^{-i\varphi(\lambda)}z + z_0, z_0)\chi_\delta(z) d\mathrm{Re}z\, d\mathrm{Im}z, \\ R_{z_0,\delta}(\lambda) &= \int_D e_{\lambda,z_0}(z)\left(v(z) - v_{lin}(z,z_0)\chi_\delta(z-z_0)\right) d\mathrm{Re}z\, d\mathrm{Im}z \end{aligned}$$

where $\varphi(\lambda) = \frac{1}{2}(\arg(\lambda) - \frac{\pi}{2})$, $e_{\lambda,z_0}(z) = e^{\lambda(z-z_0)^2 - \bar\lambda(\bar z - \bar z_0)^2}$, $z_0 \in D\setminus(\partial D)_\delta$, $\lambda\in\mathbb{C}$.

Using the stationary phase method and the explicit construction of χ_δ we obtain that

$$v(z_0) = \frac{2}{\pi}\lim_{\lambda\to\infty}|\lambda| S_{z_0,\delta}(\lambda), \tag{6.4}$$

$$|v(z_0) - \frac{2}{\pi}|\lambda| S_{z_0,\delta}(\lambda)| \le \frac{\rho_1(D)}{\delta^4}\|v\|_{C^1(\bar D)}\|\chi\|_{C^4(\mathbb{C})}|\lambda|^{-1}, \tag{6.5}$$

$z_0 \in D\setminus(\partial D)_\delta$, $0<\delta<1$, $\lambda\in\mathbb{C}$, $|\lambda|\ge 1$. Inequality (6.5) follows from

$$\begin{aligned}|v(z_0) - \frac{2}{\pi}|\lambda|S_{z_0,\delta}(\lambda)| &\le \frac{\rho_1(D)}{|\lambda|}\|v_{lin}\|_{C^4(\bar D)}\|\chi_\delta\|_{C^4(\mathbb{C})}\\ &\le \frac{\rho_1(D)}{|\lambda|\delta^4}\|v\|_{C^1(\bar D)}\|\chi\|_{C^4(\mathbb{C})},\end{aligned}$$

where we used [5, Lemma 7.7.3] and (6.3).

Integrating by parts we can write

$$\begin{aligned}R_{z_0,\delta}(\lambda) &= -\frac{1}{2\bar\lambda}\int_D \frac{\partial}{\partial \bar z}\left(e_{\lambda,z_0}(z)\right)\frac{(v(z) - v_{lin}(z,z_0)\chi_\delta(z-z_0))}{\bar z - \bar z_0} d\mathrm{Re}z\, d\mathrm{Im}z\\ &= R^1_{z_0,\delta}(\lambda) + R^2_{z_0,\delta}(\lambda),\\ R^1_{z_0,\delta}(\lambda) &= \frac{-1}{4i\bar\lambda}\int_{\partial D} e_{\lambda,z_0}(z)\frac{(v(z) - v_{lin}(z,z_0)\chi_\delta(z-z_0))}{\bar z - \bar z_0}dz,\\ R^2_{z_0,\delta}(\lambda) &= \frac{1}{2\bar\lambda}\int_D e_{\lambda,z_0}(z)\frac{\partial}{\partial \bar z}\left(\frac{(v(z) - v_{lin}(z,z_0)\chi_\delta(z-z_0))}{\bar z - \bar z_0}\right) d\mathrm{Re}z\, d\mathrm{Im}z,\end{aligned}$$

for $z_0 \in D \setminus (\partial D)_\delta$, $\lambda \in \mathbb{C}\setminus\{0\}$. In addition, we have that

$$\frac{2}{\pi}|\lambda||R^1_{z_0,\delta}(\lambda)| \le \kappa_2(D)\log(3+\delta^{-1})\|v\|_{C(\partial D)}. \tag{6.6}$$

Formula (6.6) follows from the fact that $\chi_\delta(z - z_0) = 0$ for $z \in \partial D$, $z_0 \in D \setminus (\partial D)_\delta$ and from the estimate

$$\frac{2}{\pi}|R^1_{z_0,\delta}(\lambda)| \le \frac{2}{\pi}\frac{1}{|\lambda|}\int_{\partial D}\frac{|v(z)|}{|\bar z - \bar z_0|}|dz| \le \frac{\kappa_2(D)\log(3+\delta^{-1})}{|\lambda|}\|v\|_{C(\partial D)}.$$

We now write $R^2_{z_0,\delta}(\lambda) = \frac{1}{2\bar\lambda}(R^{2,1}_{z_0,\delta,\varepsilon}(\lambda) + R^{2,2}_{z_0,\delta,\varepsilon}(\lambda))$, with

$$R^{2,1}_{z_0,\delta,\varepsilon}(\lambda) = \int_{B_{z_0,\varepsilon}} e_{\lambda,z_0}(z)\frac{\partial}{\partial \bar z}\left(\frac{(v(z) - v_{lin}(z,z_0)\chi_\delta(z-z_0))}{\bar z - \bar z_0}\right) d\mathrm{Re}z\, d\mathrm{Im}z \tag{6.7}$$

$$R^{2,2}_{z_0,\delta,\varepsilon}(\lambda) = \int_{D_{z_0,\varepsilon}} e_{\lambda,z_0}(z)\frac{\partial}{\partial \bar z}\left(\frac{(v(z) - v_{lin}(z,z_0)\chi_\delta(z-z_0))}{\bar z - \bar z_0}\right) d\mathrm{Re}z\, d\mathrm{Im}z, \tag{6.8}$$

where $B_{z_0,\varepsilon} = \{z \in \mathbb{C} : |z - z_0| < \varepsilon\}$, $D_{z_0,\varepsilon} = D \setminus B_{z_0,\varepsilon}$. In (6.7)-(6.8) we assume that $z_0 \in D \setminus (\partial D)_\delta$, $0 < \varepsilon < \delta$, $\lambda \in \mathbb{C}$.

The final part of the proof of estimate (6.2) consists in the following. We have, for $\varepsilon < \delta/2$,

$$|R^{2,1}_{z_0,\delta,\varepsilon}(\lambda)| \le \frac{7}{2}\pi\|v\|_{C^2(\bar D)}\varepsilon^2, \tag{6.9}$$

exactly as in (5.25),

$$R^{2,2}_{z_0,\delta,\varepsilon}(\lambda) = -\frac{1}{2\bar\lambda}\int_{D_{z_0,\varepsilon}} \frac{\partial}{\partial \bar z}\left(e_{\lambda,z_0}(z)\right)\frac{1}{\bar z-\bar z_0}$$
$$\times \frac{\partial}{\partial \bar z}\left(\frac{(v(z)-v_{lin}(z,z_0)\chi_\delta(z-z_0))}{\bar z-\bar z_0}\right) d\mathrm{Re}z\, d\mathrm{Im}z$$
$$= -\frac{1}{2\bar\lambda}(R^{2,2,1}_{z_0,\delta,\varepsilon}(\lambda)+R^{2,2,2}_{z_0,\delta,\varepsilon}(\lambda)),$$

$$R^{2,2,1}_{z_0,\delta,\varepsilon}(\lambda) = \frac{1}{2i}\int_{\partial D_{z_0,\varepsilon}} e_{\lambda,z_0}(z)\frac{1}{\bar z-\bar z_0}\frac{\partial}{\partial \bar z}\left(\frac{(v(z)-v_{lin}(z,z_0)\chi_\delta(z-z_0))}{\bar z-\bar z_0}\right) dz$$
$$= -\frac{1}{2i}\int_{\partial B_{z_0,\varepsilon}} e_{\lambda,z_0}(z)\frac{1}{\bar z-\bar z_0}\frac{\partial}{\partial \bar z}\left(\frac{v(z)-v_{lin}(z,z_0)}{\bar z-\bar z_0}\right) dz$$
$$-\frac{1}{2i}\int_{\partial D} e_{\lambda,z_0}(z)\frac{1}{\bar z-\bar z_0}\frac{\partial}{\partial \bar z}\left(\frac{v(z)}{\bar z-\bar z_0}\right) dz,$$

$$R^{2,2,2}_{z_0,\delta,\varepsilon}(\lambda) = -\int_{D_{z_0,\varepsilon}} e_{\lambda,z_0}(z)$$
$$\times \frac{\partial}{\partial \bar z}\left(\frac{1}{\bar z-\bar z_0}\frac{\partial}{\partial \bar z}\left(\frac{(v(z)-v_{lin}(z,z_0)\chi_\delta(z-z_0))}{\bar z-\bar z_0}\right)\right) d\mathrm{Re}z\, d\mathrm{Im}z.$$

We have, for $\varepsilon < \delta/2$

(6.10)
$$|R^{2,2,1}_{z_0,\delta,\varepsilon}(\lambda)| \le \frac12\int_{\partial B_{z_0,\varepsilon}}\frac{|v(z)-v_{lin}(z,z_0)|}{|z-z_0|^3}|dz| + \frac12\int_{\partial B_{z_0,\varepsilon}}\frac{|v_{\bar z}(z)-v_{\bar z}(z_0)|}{|z-z_0|^2}|dz|$$
$$+\frac12\int_{\partial D}\frac{|v(z)|}{|z-z_0|^3}|dz| + \frac12\int_{\partial D}\frac{|v_{\bar z}(z)|}{|z-z_0|^2}|dz|$$
$$\le \frac72\pi\|v\|_{C^2(\bar D)} + \frac{\rho_2(D)}{\delta^2}\|v\|_{C^1(\bar D)},$$

(6.11)
$$|R^{2,2,2}_{z_0,\delta,\varepsilon}(\lambda)| \le |R^{2,2,2}_{z_0,\delta,\delta/2}(\lambda)| + |R^{2,2,2}_{z_0,\delta,\varepsilon}(\lambda)-R^{2,2,2}_{z_0,\delta,\delta/2}(\lambda)|,$$

(6.12)
$$|R^{2,2,2}_{z_0,\delta,\delta/2}(\lambda)| \le \frac{\rho_3(D)}{\delta^3}\|v\|_{C^2(\bar D)},$$
$$|R^{2,2,2}_{z_0,\delta,\varepsilon}(\lambda)-R^{2,2,2}_{z_0,\delta,\delta/2}(\lambda)| \le \sum_{j=1}^{5}\int_{B_{z_0,\delta/2}\setminus B_{z_0,\varepsilon}} u_j(z,z_0)d\mathrm{Re}z\, d\mathrm{Im}z,$$

with u_j defined as in (5.29)-(5.33). This yields

$$|R^{2,2,2}_{z_0,\delta,\varepsilon}(\lambda)-R^{2,2,2}_{z_0,\delta,\delta/2}(\lambda)| \le \rho_4(D)\log(\frac{\delta}{2\varepsilon})\|v\|_{C^2(\bar D)}, \quad (6.13)$$

where $z_0 \in D \setminus (\partial D)_\delta$, $0 < \varepsilon < \delta/2$, $\lambda \in \mathbb{C} \setminus \{0\}$. Using (6.5), (6.6), (6.9)-(6.13) with $\varepsilon = |\lambda|^{-1}$ we obtain (6.2) for $|\lambda| > \frac{2}{\delta}$.

Notice that only the estimation of $|\lambda||R^2_{z_0,\delta}(\lambda)|$ requires $|\lambda| > \frac{2}{\delta}$. In that case one has

$$\frac{2}{\pi}|\lambda||R^2_{z_0,\delta}(\lambda)| \leq \rho_5(D)\delta^{-4}\frac{\log(3|\lambda|)}{|\lambda|}\|v\|_{C^2(\bar{D})}.$$

If $1 \leq |\lambda| \leq \frac{2}{\delta}$ we have that

$$\frac{2}{\pi}|\lambda||R^2_{z_0,\delta}(\lambda)| \leq \frac{\rho_6(D)N}{\delta} \tag{6.14}$$

and

$$\rho_5(D)\delta^{-4}\frac{\log(3|\lambda|)}{|\lambda|}\|v\|_{C^2(\bar{D})} \geq \frac{\rho_5(D)}{2\delta^3}\log(6\delta^{-1})\|v\|_{C^2(\bar{D})}, \tag{6.15}$$

where we used the fact that the function $\frac{\log(3s)}{s}$ is decreasing for $s > \frac{e}{3}$.

We now define

$$c' = \frac{2\rho_6(D)N}{\rho_5(D)\log(6)\|v\|_{C^2(\bar{D})}},$$

in order to have

$$\frac{2}{\pi}|\lambda||R^2_{z_0,\delta}(\lambda)| \leq c'\rho_5(D)\delta^{-4}\frac{\log(3|\lambda|)}{|\lambda|}\|v\|_{C^2(\bar{D})},$$

for $1 \leq |\lambda| \leq \frac{2}{\delta}$, $0 < \delta < 1$.

Thus, taking $\kappa_1 = \max(\rho_5, c'\rho_5, \rho_1\|\chi\|_{C^4(\mathbb{C})})$, we obtain estimation (6.2) for $|\lambda| \geq 1$ and $0 < \delta < 1$. This finish the proof of Lemma 6.2. □

Proof of Proposition 6.1. Fix $0 < \alpha < \frac{1}{5}$, and $0 < \delta < 1$. We have the following chain of inequalities

$$\begin{aligned}
\|v_2 - v_1\|_{L^\infty(D)} &= \max(\|v_2 - v_1\|_{L^\infty(D\cap(\partial D)_\delta)}, \|v_2 - v_1\|_{L^\infty(D\setminus(\partial D)_\delta)}) \\
&\leq C_1 \max\left(2N\delta + \|\Phi_2 - \Phi_1\|_1, \frac{\log(3\log(3 + \|\Phi_2 - \Phi_1\|^{-1}))}{\delta^4\log(3 + \|\Phi_2 - \Phi_1\|^{-1})}\right. \\
&\qquad \left. + \log(3 + \frac{1}{\delta})\|\Phi_2 - \Phi_1\|_1 + \frac{\log(3\log(3 + \|\Phi_2 - \Phi_1\|^{-1}))}{(\log(3 + \|\Phi_2 - \Phi_1\|^{-1}))^{\frac{1}{2}}}\right) \\
&\leq C_2 \max\left(2N\delta + \|\Phi_2 - \Phi_1\|_1, \frac{1}{\delta^4}\log(3 + \|\Phi_2 - \Phi_1\|_1^{-1})^{-5\alpha}\right. \\
&\qquad \left. + \log(3 + \frac{1}{\delta})\|\Phi_2 - \Phi_1\|_1 + \frac{\log(3\log(3 + \|\Phi_2 - \Phi_1\|_1^{-1}))}{(\log(3 + \|\Phi_2 - \Phi_1\|_1^{-1}))^{\frac{1}{2}}}\right),
\end{aligned}$$

where we followed the scheme of the proof of Theorem 1.1 with the following modifications: we make use of Lemma 6.2 instead of Lemma 3.2 and we also use i)-ii); note that $C_1 = C_1(D, N)$ and $C_2 = C_2(D, N, \alpha)$.

Putting $\delta = \log(3 + \|\Phi_2 - \Phi_1\|_1^{-1})^{-\alpha}$ we obtain the desired inequality

$$\|v_2 - v_1\|_{L^\infty(D)} \le C_3 \log(3 + \|\Phi_2 - \Phi_1\|_1^{-1})^{-\alpha}, \tag{6.16}$$

with $C_3 = C_3(D, N, \alpha)$, $\|\Phi_2 - \Phi_1\|_1 = \varepsilon \le \varepsilon_1(D, N, \alpha)$ with ε_1 sufficiently small or, more precisely when $\delta_1 = \log(3 + \varepsilon_1^{-1})^{-\alpha}$ satisfies:

$$\delta_1 < 1, \qquad \varepsilon_1 \le 2N\delta_1, \qquad \log(3 + \frac{1}{\delta_1})\varepsilon_1 \le \delta_1.$$

Estimate (6.16) for general ε (with modified C_3) follows from (6.16) for $\varepsilon \le \varepsilon_1(D, N, \alpha)$ and the assumption that $\|v_j\|_{L^\infty(\bar{D})} \le N$ for $j = 1, 2$. This completes the proof of Proposition 6.1. □

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